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# NON-PERTURBATIVE FERMION PROPAGATOR FOR THE MASSLESS QUENCHED QED3

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## ABSTRACT

For massless quenched QED in three dimensions, we evaluate a non-perturbative expression for the fermion propagator which agrees with its two loop perturbative expansion in the weak coupling regime. This calculation is carried out by making use of the Landau-Khalatnikov-Fradkin transformations. Any improved construction of the fermion-boson vertex must make sure that the solution of the Schwinger-Dyson equation for the fermion propagator reproduces this result. For two different gauges, we plot the fermion propagator against momentum. We then make a comparison with a similar plot, using the earlier expression for the fermion propagator, which takes into account only the one loop result.

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# 1 Introduction

A natural starting point for the non-perturbative study of gauge theories is the corresponding set of Schwinger-Dyson equations (SDEs). QED in 3-dimensions (QED3) has been a popular choice for such a study due to its relative simplicity and its confining behaviour in the quenched approximation. It requires knowledge of the non-perturbative form of the fundamental fermion-boson interaction. In the quenched approximation, one can then calculate the fermion propagator. Both the propagator and the vertex must obey essential gauge dependence in accordance with Landau-Khalatnikov-Fradkin (LKF) transformations [1, 2]. These transformations are written in the coordinate space representation and they allow us to evaluate a non-perturbative expression for a Greens function in an arbitrary covariant gauge if we know its value in any particular gauge. It seems an insurmountable task to know a Greens function in any gauge. However, progress can be made in perturbation theory by calculating it to a certain order.

Due to the complicated nature of the LKF transformations, it has been difficult to derive analytical conclusions for the vertex. However, the fermion propagator is relatively easier to analyze and related analytical results exist for QED in 3 and 4 dimensions, based upon the knowledge of the fermion propagator at the one loop order. These results are generally assumed to be true to all orders [3, 4] (referred to as the *transversality condition* in [5]), and constraints are derived on the non-perturbative form of the fermion-boson vertex. Realizing that this condition would not hold to all orders, Bashir *et. al.* have derived constraints on the vertex in QED4 by demanding general constraints from the multiplicative renormalizability of the fermion propagator [6]. Recently, a two loop calculation has been done for the fermion propagator in the massless quenched QED3 [7, 8], showing explicitly that the transversality condition is violated. Exploiting this calculation, we go beyond one loop and present the evaluation of the non-perturbative fermion propagator through the use of LKF transformations. In comparison with the corresponding expression in [3], our result has an added piece which makes sure that in the weak coupling regime, correct two loop behaviour of the fermion propagator is achieved. Our calculation calls for the need to construct an improved vertex which would reproduce

our result when used in the corresponding SDE. We also plot these two expressions as a function of fermion momentum and compare the results for two different gauges.

## 2 Fermion-Boson Vertex and Gauge Invariance

The study of the fermion propagator in quenched QED requires making an *ansatz* for the vertex. An acceptable *ansatz* must ensure the inclusion of the key features required of it. We shall only focus on the features relevant to the discussion in this paper:

- The vertex  $\Gamma^\mu(k, p)$  must satisfy the Ward-Green-Takahashi Identity (WGTI) which relates it to the fermion propagator  $S_F(p^2)$ :

$$q_\mu \Gamma^\mu(k, p) = S_F^{-1}(k) - S_F^{-1}(p), \quad (1)$$

where  $q = k - p$ .

- It must reduce to the Feynman expansion of the perturbative vertex in the weak coupling regime.
- It must ensure local gauge covariance of the propagators and vertices.

Although the WGTI is a consequence of gauge invariance, it only fixes the longitudinal part  $\Gamma_L(k, p)$  of the complete vertex [9],  $\Gamma^\mu(k, p) = \Gamma_L^\mu(k, p) + \Gamma_T^\mu(k, p)$ , whereas the transverse part  $\Gamma_T(k, p)$ , defined by the equation  $q_\mu \Gamma_T^\mu(k, p) = 0$ , remains undetermined. Without a proper choice of this part, one cannot ensure the local gauge covariance of the propagators and the vertex as demand the LKF transformations. Unfortunately, the LKF transformation law for the vertex is too complicated to be made use of. However, the corresponding rule for the fermion propagator is relatively simple. A proper choice of the transverse vertex in an arbitrary gauge is essential to satisfy it, as it is related to the fermion propagator through the following SDE in the Euclidean space, Fig. (1):

$$S_F^{-1}(p) = S_F^{0-1}(p) + e^2 \int \frac{d^3 k}{(2\pi)^3} \Gamma^\mu(k, p) S_F(k) \gamma^\nu \Delta_{\mu\nu}^0(q), \quad (2)$$

where  $S_F^0(p) = 1/i \not{p}$  and we express  $S_F(p) = F(p^2)/i \not{p}$ . The photon propagator can be split into the transverse and the longitudinal parts as:

$$\Delta_{\mu\nu}^0(q) = \Delta_{\mu\nu}^{0T}(q) + \xi \frac{q_\mu q_\nu}{q^4}, \quad (3)$$

where  $\Delta_{\mu\nu}^{0\ T}(q) = [\delta_{\mu\nu} - q_\mu q_\nu / q^2] / q^2$ . Burden and Roberts [3] pointed out that the condition

$$\int \frac{d^3k}{(2\pi)^3} \Gamma^\mu(k, p) S_F(k) \gamma^\nu \Delta_{\mu\nu}^{0\ T}(q) = 0, \quad (4)$$

the so called transversality condition [5], leads to the correct LKF behaviour of the fermion propagator. This condition ensures that  $F(p^2) = 1$  in the Landau gauge. The LKF transformations then yield

$$F(p^2) = 1 - \frac{\alpha\xi}{2p} \tan^{-1} \left[ \frac{2p}{\alpha\xi} \right]. \quad (5)$$

However, it has been shown in [7, 8] that although this condition is satisfied at one loop order, it gets violated at the two loop order, leading to the following expression for the fermion propagator <sup>1</sup>:

$$F(p^2) = 1 - \frac{\pi\alpha\xi}{4p} + \frac{\alpha^2\xi^2}{4p^2} - \frac{3\alpha^2}{4p^2} \left( \frac{7}{3} - \frac{\pi^2}{4} \right) + \mathcal{O}(\alpha^3). \quad (6)$$

This expression for the fermion propagator of course does not satisfy the LKF transformations in a non-perturbative fashion. However, making use of the said transformations, we can calculate an expression for the fermion propagator which does transform non-perturbatively as require the LKF transformations, which is an important requirement of gauge covariance. We carry out this exercise in the next section.

### 3 Propagator and the LKF transformation

Let us use the following notation and definition of the massless fermion propagator in the momentum and coordinate spaces respectively, in an arbitrary covariant gauge  $\xi$ :

$$\begin{aligned} S_F(p; \xi) &= \frac{F(p; \xi)}{i \not{p}}, \\ S_F(x; \xi) &= \not{x} X(x; \xi). \end{aligned} \quad (7)$$

These expressions are related by the following Fourier transforms:

$$\begin{aligned} S_F(p; \xi) &= \int d^3x e^{ip \cdot x} S_F(x; \xi) \\ S_F(x; \xi) &= \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot x} S_F(p; \xi). \end{aligned} \quad (8)$$

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<sup>1</sup>An error made in the first reference of [7] was corrected in the second and in [8]

The LKF transformation relating the coordinate space fermion propagator in the Landau gauge to the coordinate space fermion propagator in an arbitrary covariant gauge reads:

$$S_F(x; \xi) = S_F(x; 0) e^{-(\alpha\xi/a)x} . \quad (9)$$

The following are the steps to find the non-perturbative expression for the fermion propagator in momentum space in an arbitrary covariant gauge: (i) Input the perturbative expression for the fermion propagator in the Landau gauge, i.e.,  $F(p; 0)$ . (ii) Evaluate  $X(x; 0)$  by taking the Fourier transform. (iii) Calculate  $X(x; \xi)$  by using the LKF transformation law. (iv) Fourier transform back the result to  $F(p; \xi)$ . Eq. (6) implies that

$$F(p; 0) = a_0 + a_1 \frac{\alpha}{p} + a_2 \frac{\alpha^2}{p^2} + \mathcal{O}(\alpha^3) , \quad (10)$$

where  $a_0 = 1$ ,  $a_1 = 0$  and  $a_2 = -7/4 + 3\pi^2/16$ . Although  $a_1 = 0$ , we shall keep this term in order to prove a point later. Eq. (8) permits us to carry out the Fourier transform of Eq. (10). On doing that and carrying out the angular integration, we get

$$X(x; 0) = - \sum_{n=0}^{n=2} \frac{a_n \alpha^n}{2\pi^2 x^3} \int_0^\infty \frac{dp}{p^{n+1}} (\sin px - px \cos px) .$$

The radial integration then yields:

$$X(x; 0) = -\frac{a_0}{4\pi} \frac{1}{x^3} - \frac{a_1}{8\pi} \frac{\alpha}{x^2} - \frac{a_2}{8\pi} \frac{\alpha^2}{x} . \quad (11)$$

This result in the Landau gauge is related to that in arbitrary covariant gauge through the LKF transformation, Eq. (9). In order to Fourier transform the result back to the momentum space, we use Eq. (8). Substitute Eq. (9) in it, multiply the equation by  $\not{p}$  and then take the trace:

$$F(p; \xi) = i \int d^3x \not{p} \cdot x e^{ip \cdot x} X(x; 0) e^{-(\alpha\xi/2)x} . \quad (12)$$

Substituting Eq. (11) in it and carrying out the integrations, we obtain:

$$F(p; \xi) = a_0 - \frac{\alpha(\pi\xi - 4a_1)}{2\pi p} \tan^{-1} \frac{2p}{\alpha\xi} - \frac{4\alpha^2}{\alpha^2\xi^2 + 4p^2} \left[ \frac{a_1\xi}{\pi} - \frac{4a_2p^2}{\alpha^2\xi^2 + 4p^2} \right] . \quad (13)$$

Expanding this expression around small values of  $\alpha$  we get

$$F(p, \xi) = a_0 + a_1 \frac{\alpha}{p} - \frac{\pi\xi\alpha}{4p} - a_1 \frac{\alpha^2\xi}{\pi p^2} + a_2 \frac{\alpha^2}{p^2} + \frac{\alpha^2\xi^2}{4p^2} + \mathcal{O}(\alpha^4) . \quad (14)$$

For  $\xi = 0$ , we recuperate Eq. (10). There are some interesting points to note:

- The constant  $a_1$  appears as a coefficient of  $\mathcal{O}(\alpha)$  term as well as  $\mathcal{O}(\alpha^2\xi)$  term in Eq. (14). The fact that  $a_1 = 0$  thus automatically rules out the presence of  $\mathcal{O}(\alpha^2\xi)$  term.
- Apart from the  $a_1$  terms,  $\xi$  always appears in conjunction with  $\alpha$ , i.e., in the form  $\alpha\xi$ .
- In the perturbative expression, Eq. (14), there exists no  $\alpha^3$  term, and the same is true for higher odd powers of  $\alpha$ . This does not of course rule out the possibility of encountering these terms on perturbative evaluation of  $F(p, \xi)$  at the three-loop level, and so on.

Substituting the values of  $a_0$ ,  $a_1$  and  $a_2$ , we arrive at the following final result:

$$F(p; \xi) = 1 - \frac{\alpha\xi}{2p} \tan^{-1} \frac{2p}{\alpha\xi} - \frac{(28 - 3\pi^2)p^2\alpha^2}{(\alpha^2\xi^2 + 4p^2)^2}. \quad (15)$$

This expression contains an expected additional term in comparison with Eq. (5). This term ensures that the perturbative expansion of Eq. (15) matches correctly on to the two-loop calculation of  $F(p; \xi)$ . It also has the correct gauge dependence non-perturbatively as demanded by LKF transformations, in contrast with Eq. (6). Moreover, being non-perturbative in nature, it contains exact information of terms of orders higher than  $\alpha^2$ . For example, the perturbative expansion of Eq. (15) to  $\mathcal{O}(\alpha^4)$  reads:

$$\begin{aligned} F(p^2) = & 1 - \frac{\pi\alpha\xi}{4p} + \frac{\alpha^2\xi^2}{4p^2} - \frac{3\alpha^2}{4p^2} \left( \frac{7}{3} - \frac{\pi^2}{4} \right) - \frac{\alpha^4\xi^4}{48p^4} \\ & - \frac{3\alpha^4\xi^2}{8p^4} \left( \frac{7}{3} - \frac{\pi^2}{4} \right) + \mathcal{O}(\alpha^6). \end{aligned} \quad (16)$$

Note that there are no  $\alpha^3$  terms in this expression. Therefore, in conjunction with the structure of LKF transformation, we conclude that in the actual perturbative calculation of  $F(p^2)$  to  $\mathcal{O}(\alpha^3)$ , there will be no terms of the type  $\alpha^3\xi^3$ ,  $\alpha^3\xi^2$ , or  $\alpha^3\xi$ . This information is not contained in Eq. (6), which of course does not know anything about orders higher than  $\alpha^2$ . Similarly for  $\mathcal{O}(\alpha^4)$ , Eq. (15) exactly gives the coefficients of  $\alpha^4\xi^4$ ,  $\alpha^4\xi^3$  and  $\alpha^4\xi^2$  terms in perturbation theory, and so on for higher order terms.

In perturbation theory, every higher order term is expected to be much smaller than the term in the previous order in a systematic way. Naturally, one wonders what is the relative contribution of the additional piece in Eq. (15) in the non-perturbative regime. In Figs. (2,3), we have drawn  $F(p^2)$  as obtained from Eq. (5) and Eq. (15) for two different values of the gauge parameter. For larger values of the gauge parameter, the two results start merging into each other. However, for low values of the gauge parameter, a bump arises in the  $F(p^2)$  given by Eq. (15) at low values of  $p$  because of the maximum in the additional piece at  $p = \alpha\xi/2$ . Therefore, one concludes that for higher values of the gauge parameters, the additional piece modifies Eq.(5) insignificantly. However, for decreasing values of the gauge parameter, the difference starts increasing for low momenta. In essence, Figs. (2,3) display the values of the gauge parameter for which the more complete Eq. (15) will deviate significantly from Eq. (5) in the non-perturbative regime.

## 4 Conclusions

In this paper, we present the calculation of the non-perturbative fermion propagator using the knowledge of its two-loop expansion and the LKF transformations. It is natural to assume that physically meaningful solutions of the Schwinger-Dyson equations must agree with perturbative results in the weak coupling regime. This realization has been used in QED4 [6, 10, 11], and more recently in QED3 [7, 8], to use perturbation theory as a guide towards the non-perturbation truncation of Shwinger-Dyson equations. So far, progress has been made in this context by attempting to make the correspondence of non-perturbative propagators and vertex to their one-loop expansion. In this paper, we have gone beyond the one-loop order and constructed a fermion propagator which agrees with perturbation theory at least up to two-loops, and also has the correct gauge dependence as demanded by its LKF transformations. On the numerical side, it is important to know the contribution of the new piece in Eq. (15) as compared to the rest of the equation. It turns out that for higher values of the gauge parameter we do not need to worry about it. However in the neighbourhood of the Landau gauge, it causes significant deviation of the total result in comparison with the one in its absence for low values of momentum.

As the fermion propagator is related to the 3-point vertex through its SDE, our results put constraints on the possible forms for the unknown transverse part of the vertex. This part is in principle determined by understanding how the essential gauge dependence of the vertex demanded by its LKF transformation is satisfied non-perturbatively. In practice it is not an easy condition to implement. However, a simpler constraint is that any non-perturbative construction of the transverse vertex must ensure that we recuperate Eq. (15) when used in the SDE for the fermion propagator, leading to a more reliable non-perturbative truncation of SDEs.

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# Figure Captions

1. Schwinger-Dyson equation for fermion propagator in quenched QED.
2.  $F(p : \xi)$  as a function of  $p$  for  $\xi = 1$ . The dotted and the continuous lines correspond to Eq. (5) and Eq. (15) respectively.
3.  $F(p : \xi)$  as a function of  $p$  for  $\xi = 0.5$ . The dotted and the continuous lines correspond to Eq. (5) and Eq. (15) respectively.

## Figures

Fig. 1

$$\begin{array}{c}
 \text{---} \bullet \text{---} \\
 \downarrow \vec{p}
 \end{array}
 \overset{-1}{=}
 \begin{array}{c}
 \text{---} \\
 \downarrow \vec{p}
 \end{array}
 \overset{-1}{-}
 \begin{array}{c}
 \begin{array}{c} \text{q} \\ \leftarrow \end{array} \\
 \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \\ \downarrow \vec{k} \end{array}
 \end{array}$$

Fig. 2

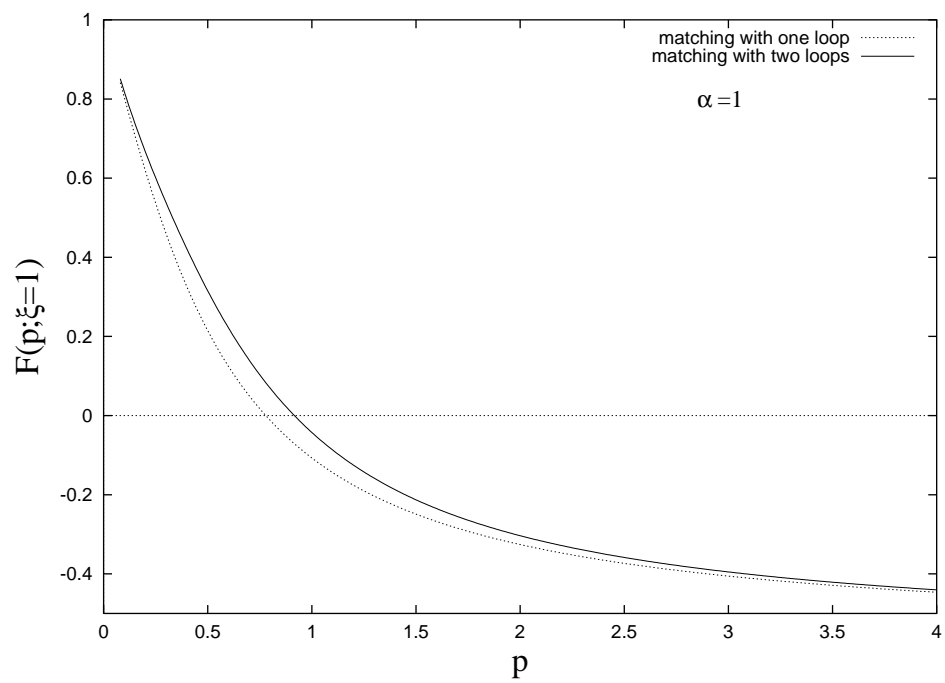


Fig. 3

